exact, be eliminated. ... unchanged and the rest will, in

EXAMPLE 6.10-1 (Assignment Problem)

There are n jobs to be manufactured and n machines are available. Each job can be processed on each of the machines. The cost of processing is represented by the table below. If

TABLE 6.127

		Jobs			
		1	2	...	n
Machines	1	c_{11}	c_{12}	...	c_{1n}
	2	c_{21}	c_{22}	...	c_{2n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n	c_{n1}	c_{n2}	...	c_{nn}

one job is to be processed on one machine only, the problem is to determine the optimal assignment policy that will turn out all the jobs at minimum total cost. Formulate this problem as I.P. model.

Formulation as I.P. Problem

Let $$x_{ij} = \begin{cases} 1, \text{ if } i\text{th machine is assigned the } j\text{th job,} \\ 0, \text{ if } i\text{th machine is not assigned the } j\text{th job.} \end{cases}$$

Then the I.P. model is given by

minimize $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij}\, x_{ij},$$

subject to the constrainte

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, 2, ..., n;$$

$$\sum_{i=1}^{n} x_{ij} = 1,\ j = 1, 2, ..., n.$$

EXAMPLE 6.10-2 (Travelling Salesman Problem)

A book salesman who lives at city 1 must call once a month on four customers located in cities 2, 3, 4 and 5. The following table gives the distance in kilometers among the different cities.

TABLE 6.128

To City

		1	2	3	4	5
	1	–	210	150	250	110
	2	210	–	100	80	130
From City	3	150	100	–	60	105
	4	250	80	60	–	90
	5	110	130	150	90	–

The objective is to minimize the total distance travelled by the salesman. Formulate the problem as I.P. model.

Formulation as I.P. Problem

As travelling from city i to i is not possible, a prohibitively high cost $c_{ii} = M$, $i = 1, 2, ..., 5$ is assigned to the diagonal cells.

Let $$x_{ij} = \begin{cases} 1, \text{ if the salesman travels from city } i \text{ to city } j, \\ 0, \text{ otherwise.} \end{cases}$$

The necessary condition for a complete route is that city i connects to one city only and that city j is reached from exactly one city. Then I.P. model for the problem is

minimize $$\begin{aligned} Z = [&210x_{12} + 150x_{13} + 250x_{14} + 110x_{15} + 210x_{21} + 100x_{23} + 80x_{24} \\ &+ 130x_{25} + 150x_{31} + 100x_{32} + 60x_{34} + 105x_{35} + 250x_{41} + 80x_{42} \\ &+ 60x_{43} + 90x_{45} + 110x_{51} + 130x_{52} + 150x_{53} + 90x_{54} \\ &+ M(x_{11} + x_{22} + x_{33} + x_{44} + x_{55})] \text{ km,} \end{aligned}$$

subject to
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 1,$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1,$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 1,$$
$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 1,$$
$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} = 1,$$
$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 1,$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 1,$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 1,$$
$$x_{14} + x_{24} + x_{34} + x_{44} + x_{54} = 1,$$
and
$$x_{15} + x_{25} + x_{35} + x_{45} + x_{55} = 1.$$

EXAMPLE 6.10-3 (Capital Budgeting Problem)

A company manufacturing chemicals has 4 independent investment projects and w allocate a fixed capital budget to one or more of them so that the company's total assets a maximized. The estimated investments and the anticipated cash outflows associated with th projects are given in the table below.

TABLE 6.129

Project	Investment (Rs. lakhs) 1st year	Investment (Rs. lakhs) 2nd year	Cash inflows (Rs. lakhs)
A	60	160	105
B	108	140	140
C	200	150	80
D	90	70	100

The comapny has earmarked Rs. 600 lakhs for investment in the first year and Rs. 7 lakhs in the second year. If projects A and C are mutually exclusive, how should the investm be made so that the company's total assets are maximized ?

Formulation as I.P. Problem

Let
$$x_j = \begin{cases} 1, & \text{if there is investment on the } j\text{th project} \\ 0, & \text{otherwise } (j = 1, 2, 3, 4) \end{cases}$$
for projects A, B, C, and D.

Then the objective is
$$\text{maximize } Z = 105x_1 + 140x_2 + 80x_3 + 100x_4.$$

The constraints are on availability of the funds for the two years and can be express
$$60x_1 + 108x_2 + 200x_3 + 90x_4 \leq 600,$$
and
$$160x_1 + 140x_2 + 150x_3 + 70x_4 \leq 700.$$

Since projects A and C are mutually exclusive,
$$x_1 + x_3 = 1.$$

Remark : In the above three situations variables were discrete and hence there difficulty in I.P. model formulations. Difficulty arises in situations where direct I.P. form are not possible. Codification of variables is helpful in such situations. A few situati require codification and transformation are now presented.

EXAMPLE 6.10-4 (Fixed Charge Problem)

(a) Consider a production planning problem wherein it is required to produce n it the production of the jth item involve two types of set-up costs, the fixed cost K_j independent of the quantity produced and a variable cost c_j per unit. If x_j is the number of the jth item produced, the production cost function for the jth item may be written a

$$C_j(x_j) = \begin{cases} K_j + c_j x_j, & x_j > 0, \\ 0, & x_j = 0. \end{cases}$$

The objective function then becomes

$$\text{minimize} \quad Z = \sum_{j=1}^{n} C_j(x_j)$$

$$= \sum_{j=1}^{n} (K_j + c_j x_j).$$

The corresponding constraint is

$$\sum_{j=1}^{n} x_j \geq n, \quad x_j \geq 0 \quad \text{and integer.}$$

It may be noted that the objective function is non-linear because of the fixed cost K_j involved. This difficulty can, however, be overcome by reformulation of the problem as a mixed integer programming problem with the introduction of auxiliary binary variables y_j, given by

$$y_j = \begin{cases} 1, \text{ if } x_j > 0, \\ 0, \text{ if } x_j = 0. \end{cases}$$

These conditions can be expressed as a single linear constraint

$$x_j \leq M\, y_j,$$

in which M denotes a very large number (exceeding the largest feasible value of x_j for all j). The model can, therefore, be expressed as

$$\text{minimize} \quad Z = \sum_{j=1}^{n} (K_j y_j + c_j x_j),$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_j \geq n,$$

$$x_j \leq M\, y_j,$$

$$x_j \geq 0 \text{ and integer,}$$

$$y_j = 1 \text{ or } 0 \text{ for all } j.$$

Note that although the original fixed charge problem has nothing to do with integer problem, the "transformed" problem is a zero-one mixed integer problem.

(b) *Consider the following production data :*

Product	*Profit / unit (Rs.)*	*Direct labour requirement (hours)*
1	*11*	*16*
2	*14*	*20*
3	*9*	*10*

Fixed cost (Rs.)	*Direct labour requirement (hours)*
10,000	*Up to 15,000*
18,000	*15,000 to 30,000*
25,000	*30,000 to 60,000*

Formulate an integer programming problem to determine the production schedule so as to maximize the total net profit.

Formulation as I.P. Problem

Let x_1, x_2, x_3 be the number of units of the product 1, 2 and 3 respectively to be produced. Further, let y_j, $j = 1, 2, 3$ be the binary integer variable corresponding to the fixed cost of Rs. 10,000, Rs. 18,000 and Rs. 25,000 respectively, where $y_j = 0$ or 1.

Then the integer L.P. problem can be expressed as

maximize $Z = 11x_1 + 14x_2 + 9x_3 - 10{,}000y_1 - 18{,}000y_2 - 25{,}000y_3$,

subject to the constraints

$$16x_1 + 20x_2 + 10x_3 \le 15{,}000y_1 + 30{,}000y_2 + 60{,}000y_3,$$
$$y_1 + y_2 + y_3 = 1,$$

where $x_1,\ x_2,\ x_3 \ge 0$,

$y_j = 0$ or 1, $j = 1, 2, 3$.

EXAMPLE 6.10-5 (Job Sequencing Problem)

Three products A, B and C are to be produced using four machines. The technological sequence and the processing time on the machines for the three products are shown in Fig. 6.8.

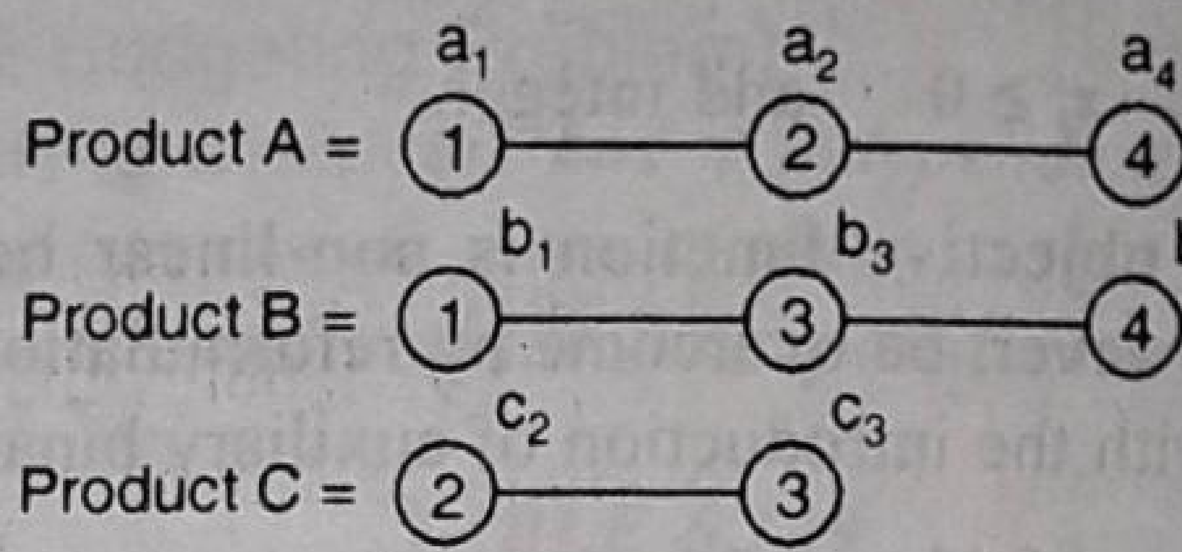

Fig. 6.8

For instance, product A is processed on machine 1 for a_1 hours, then on machine 2 for a_2 hours and finally on machine 4 for a_4 hours. Each machine processes only one product at a time and must complete its processing before taking up the next one. Further product B must be completed in not more than d hours from the start. Determine the optimal sequence in which the products be processed in order to complete all the products in the least possible time.

Formulation as I.P. Problem

Let x_{A_j} denote the time (measured from zero datum) at which the processing of product A starts on machine j ($j = 1, 2, 4$). Likewise x_{B_j} and x_{C_j} are defined.

The technological sequence in which the three products are to be machined results in the first set of constraints. Product A is processed first on machine 1, then on machine 2, and finally on machine 4. In other words

$$x_{A_1} + a_1 \le x_{A_2}$$

and $$x_{A_2} + a_2 \le x_{A_4}.$$

Similarly for products B and C, we get

$$x_{B_1} + b_1 \le x_{B_3},$$
$$x_{B_3} + b_3 \le x_{B_4},$$
$$x_{C_2} + c_2 \le x_{C_3}.$$

The next set of constraints is due to the fact that no machine can work on more than one product at a time. For example, machine 1 can process either product A or B at a given time. In other words either product A precedes B on machine 1 or vice versa. Thus we have an "either-or" type constraint for machine 1 given by

$$x_{A_1} + a_1 \le x_{B_1},$$

or $$x_{B_1} + b_1 \le x_{A_1}.$$

The above "either-or" constraint can be modified to the following two constraints with the help of a binary integer variable :

$$x_{A_1} + a_1 - x_{B_1} \le M\, y_1,$$

and $$x_{B_1} + b_1 - x_{A_1} \le (1 - y_1)\, M,$$

$$0 \le y_1 \le 1,\ y_1 \text{ integer}.$$

The first constraint becomes effective when $y_1 = 0$, implying that product A precedes B, while the second constraint becomes effective when $y_1 = 1$, implying that product B precedes A on machine 1.

Likewise, for machines 2, 3 and 4 we obtain

$$x_{A_2} + a_2 - x_{C_2} \le M\, y_2,$$
$$x_{C_2} + c_2 - x_{A_2} \le (1 - y_2)\, M,$$
$$x_{B_3} + b_3 - x_{C_3} \le M\, y_3,$$
$$x_{C_3} + c_3 - x_{B_3} \le (1 - y_3)\, M,$$
$$x_{A_4} + a_4 - x_{B_4} \le M\, y_4,$$
$$x_{B_4} + b_4 - x_{A_4} \le (1 - y_4)\, M,$$
$$0 \le y_2, y_3, y_4 \le 1,\ y_2, y_3, y_4 \text{ integers.}$$

Further, the time constraint for product B yields

$$x_{B_4} + b_4 \le d.$$

To write the objective function, note that product A will be completed at time $x_{A_4} + a_4$, product B at $x_{B_4} + b_4$ and product C at $x_{C_3} + c_3$. If Z denotes the time by which all the three products are completed, then the objective is to minimize Z, where

$$Z = \max\left(x_{A_4} + a_4,\ x_{B_4} + b_4,\ x_{C_3} + c_3\right).$$

This non-linear objective function forms a further set of three constraints

$$Z \ge x_{A_4} + a_4,$$
$$Z \ge x_{B_4} + b_4,$$
and
$$Z \ge x_{C_3} + c_3.$$

Therefore, the complete formulation of the mixed integer problem is to minimize Z subject to the above constraimts, $0 \le y_i \le 1$, y_i integer for $i = 1, 2, 3, 4$ and all other variables just non-negative.

EXAMPLE 6.10-6 (Warehouse Location Problem)

A company has plans to open two new warehouses in an area. Three sites S_1, S_2 and S_3 are under consideration. Four customers with demands D_1, D_2, D_3 and D_4 are to be supplied. Any two sites can supply all the demands but while site S_1 can supply all customers, site S_2 can supply only D_1, D_2 and D_4 and site S_3 can supply only D_2, D_3 and D_4. C_{ij} denotes the unit transportation cost from site S_i to customer D_j. The following data is given for each site :

TABLE 6.130

Site	*Capacity* (*Rs.*)	*Initial capital investment* (*Rs.*)	*Unit operating cost* (*Rs.*)
S_1	C_1	K_1	O_1
S_2	C_2	K_2	O_2
S_3	C_3	K_3	O_3

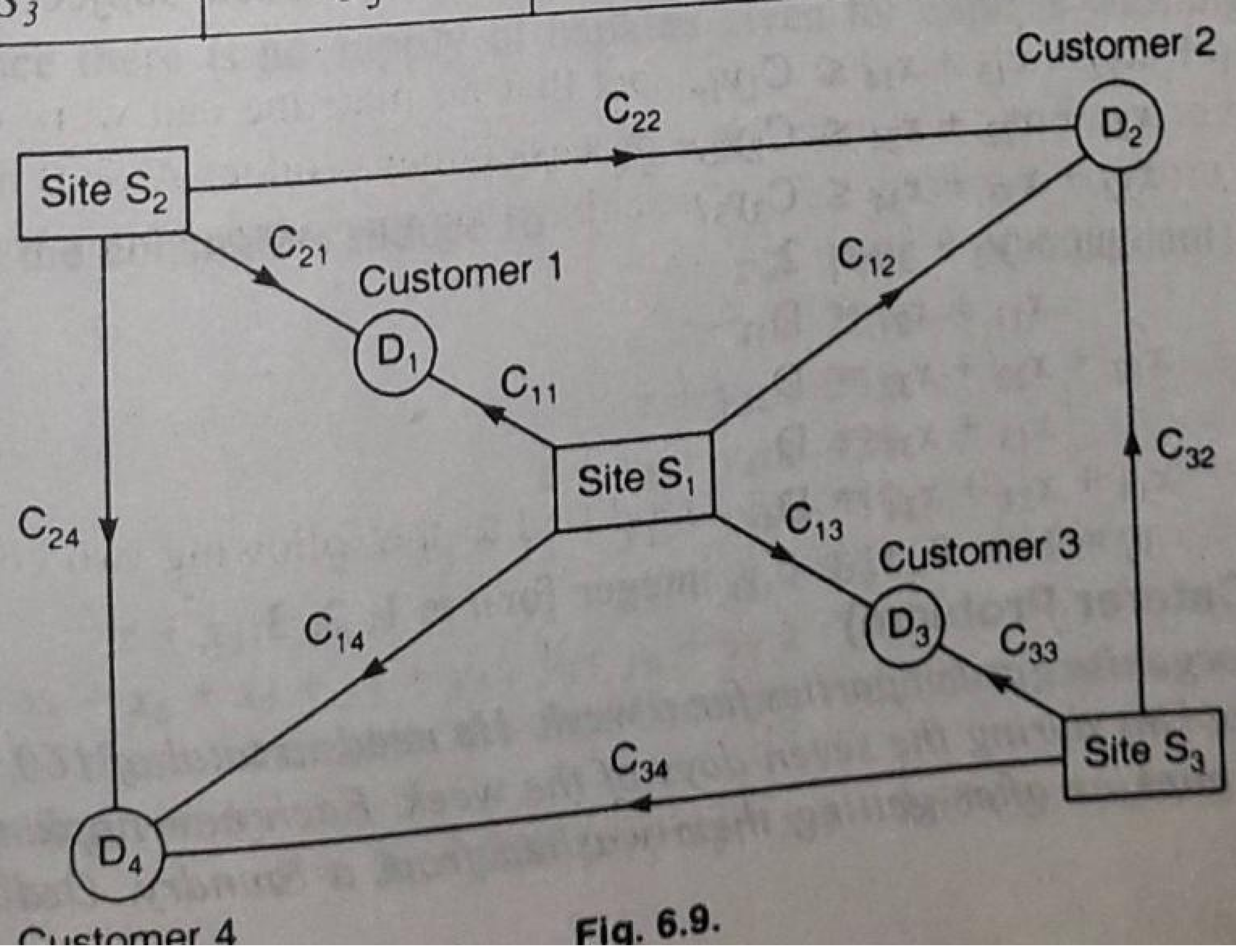

Fig. 6.9.

Select the proper sites for the two warehouses that minimize the total investment, operation and transportation costs.

Formulation as I.P. Problem

Each warehouse site has a fixed initial capital cost which is independent of the quantity stored along with variable operating and transportation costs which are proportional to the quantity shipped. The total cost function is therefore non-linear and binary integer variables can be used to formulate this problem as an I.P. problem.

If the binary integer variable y_i denotes the decision to select or not to select the site s_i, then

$$y_i = \begin{cases} 1, \text{ if site } s_i \text{ is selected,} \\ 0, \text{ if otherwise.} \end{cases}$$

If x_{ij} represents the quantity shipped from site S_i to customer D_j, then the supply constraint for site S_1 is

$$x_{11} + x_{12} + x_{13} + x_{14} \le C_1 y_1.$$

When $y_1 = 1$, site S_1 is selected and the quantity shipped cannot exceed the capacity C_1. However, when $y_1 = 0$, the non-negative variables $x_{11}, x_{12}, x_{13}, x_{14}$ will automatically be zero implying that no shipment is possible from site S_1.

Likewise, constraints for sites S_2 and S_3 are

$$x_{21} + x_{22} + x_{24} \le C_2 y_2,$$
$$x_{32} + x_{33} + x_{34} \le C_3 y_3.$$

Since only two sites are to be selected, we have the additional constraint

$$y_1 + y_2 + y_3 = 2.$$

The constraints on demand can be expressed as

$$\begin{aligned} x_{11} + x_{21} &= D_1 \quad \text{(for customer 1),} \\ x_{12} + x_{22} + x_{32} &= D_2 \quad \text{(for customer 2),} \\ x_{13} + x_{33} &= D_3 \quad \text{(for customer 3),} \\ x_{14} + x_{24} + x_{34} &= D_4 \quad \text{(for customer 4).} \end{aligned}$$

To formulate the objective function, we note that the total cost of investment, operation and transportation for site S_1 is

$K_1 y_1 + O_1 (x_{11} + x_{12} + x_{13} + x_{14}) + C_{11}x_{11} + C_{12}x_{12} + C_{13}x_{13} + C_{14}x_{14}$. The total cost function for sites S_2 and S_3 can similarly be written. Thus the objective function equation can be expressed as

$$\begin{aligned} \text{minimize} \quad Z = {} & K_1 y_1 + O_1 (x_{11} + x_{12} + x_{13} + x_{14}) + C_{11}x_{11} + C_{12}x_{12} + C_{13}x_{13} + C_{14}x_{14} \\ & + K_2 y_2 + O_2 (x_{21} + x_{22} + x_{24}) + C_{21}x_{21} + C_{22}x_{22} + C_{24}x_{24} \\ & + K_3 y_3 + O_3 (x_{32} + x_{33} + x_{34}) + C_{32}x_{32} + C_{33}x_{33} + C_{34}x_{34}. \end{aligned}$$

The above objective function is, therefore, to be minimized subject to the following constraints :

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &\le C_1 y_1, \\ x_{21} + x_{22} + x_{24} &\le C_2 y_2, \\ x_{32} + x_{33} + x_{34} &\le C_3 y_3, \\ y_1 + y_2 + y_3 &= 2, \\ x_{11} + x_{21} &= D_1, \\ x_{12} + x_{22} + x_{32} &= D_2, \\ x_{13} + x_{33} &= D_3, \\ x_{14} + x_{24} + x_{34} &= D_4, \\ y_i = 1 \text{ or } 0, \; x_{ij} \ge 0, y_i & \text{ integer for } i = 1, 2, 3. \end{aligned}$$

Example 6.10-7 (Caterer Problem)

A caterer is to organise garden parties for a week. He needs a total of 160, 120, 60, 80, 90, 100 and 105 fresh napkins during the seven days of the week. Each new napkin costs Rs. 2. He can also use soiled napkins after getting them washed from a laundry. Ordinarily, washing